Electric Circuits

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Contents

References

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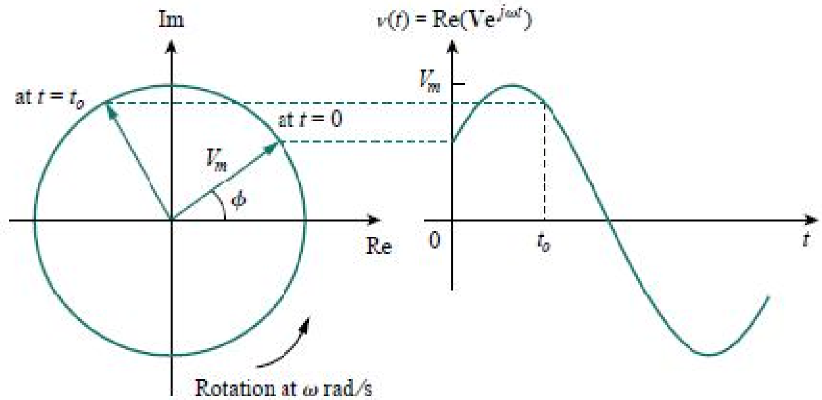
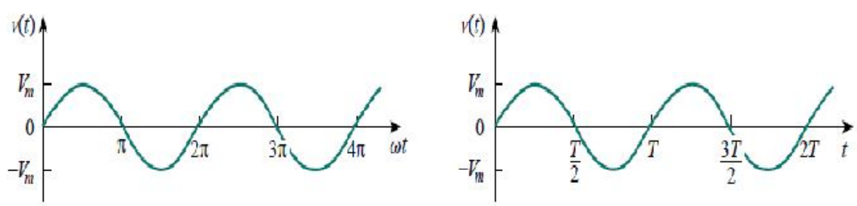
Chapter I

SinuSoidS, PhaSorS and Resonance

sinusoids



Definitions



$$v(t) = V_m \sin \omega t$$

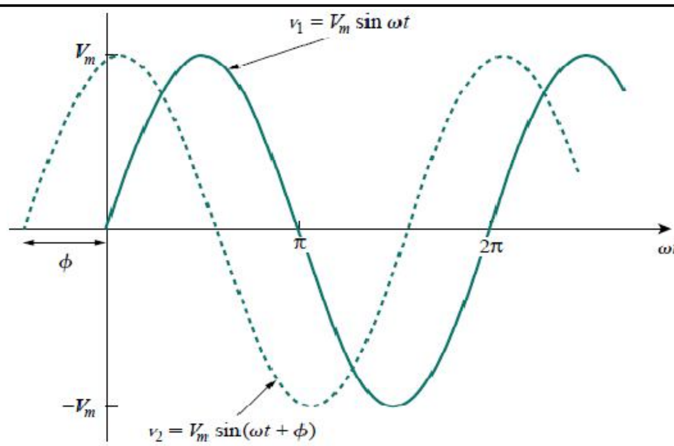
V_m = the *amplitude* of the sinusoid

ω = the *angular frequency* in radians/s

ωt = the *argument* of the sinusoid

$T = \frac{2\pi}{\omega}$ T is called the *period* of the sinusoid

$f = \frac{1}{T}$ $\omega = 2\pi f$



$$v_1(t) = V_m \sin \omega t \qquad \text{and} \qquad v_2(t) = V_m \sin(\omega t + \phi)$$

$$A \cos \omega t + B \sin \omega t = C \cos(\omega t - \theta)$$

$$C = \sqrt{A^2 + B^2}, \qquad \theta = \tan^{-1} \frac{B}{A}$$

Ex.1: **Find the amplitude, phase, period, and frequency of the sinusoid: $v(t)=12\cos(50t+10^o)$**

Solution:

The amplitude is: $V_m = 12$ V

The phase is : $\phi = 10^o$

The angular frequency is $\omega = 50$ rad/s

The period is: $T =(2\pi/\omega) = (2\pi/50) =0.1257$ s

The frequency is $f =(1/T) = (1/0.1257) = 7.958$ Hz

Ex.2 : Calculate the phase angle between
v_1 = -10 cos(ωt+50°) & v_2 = 12 sin(ωt-10°)
state which sinusoid is leading

Solution

$$v_1 = -10\cos(\omega t + 50^\circ) = 10\cos(\omega t + 50^\circ - 180^\circ)$$

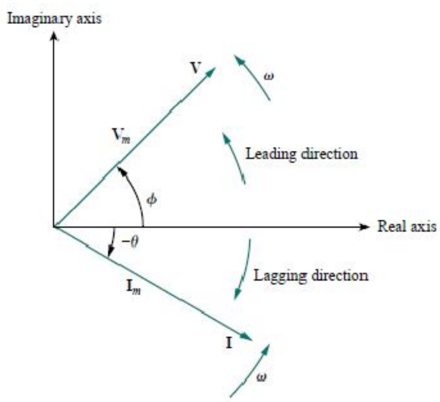
$$v_1 = 10\cos(\omega t - 130^\circ)$$

$$v_2 = 12\sin(\omega t - 10^\circ) = 12\cos(\omega t - 10^\circ - 90^\circ)$$

$$v_2 = 12\cos(\omega t \quad 100^\circ)$$

v_2 leads v_1 by 30°.

Phasor diagram



$$v(t) = V_m \sin(\omega t + \phi) = \mathrm{Im}(V_m e^{j(\omega t + \phi)})$$

$$v(t) = \mathrm{Im}(V_m e^{j\omega t} e^{j\phi}) = \mathrm{Im}(V e^{j\omega t})$$

$$V = \mathrm{Im}(V_m e^{j\phi}) = V_m e^{j\phi} = V_m \angle \phi$$

Phasors

$$V = V_m \angle \phi \qquad I = I_m \angle \theta$$

Ex. 3 : Transform these sinusoids to phasors:

(a) $v = -4\sin(30t+50^o)$

(b) $i = 6\cos(50t-40^o)$

Solution

(a) $v = -4\sin(30t+50^o) = 4\cos(30t+50+90^o)$
$= 4\cos(30t + 140^o)$

The phasor form of v is $V = 4\angle 140^o$

(b) $i = 6\cos(50t - 40^o)$

The phasor $I = 6\angle -40^o$

Ex.4: Given $i_1(t)=4\cos(\omega t+30^o)$ and $i_2(t)=5\sin(\omega t-20^o)$, find their sum.

Solution:

$i_1(t)= 4\cos(\omega t+30^o)$ $I_1 = 4\angle 30^o$

$i_2(t)= 5\sin(\omega t-20^o) = 5\cos(\omega t -20^o-90^o) = 5\cos(\omega t-110^o)$

$I_2 = 5\angle -110^o$

$I = I_1 + I_2 = 4\angle 30^o + 5\angle -110^o = 3.464+j2-1.71-j4.698$
$= 1.754 - j2.698 = 3.218\angle -56.97^o$

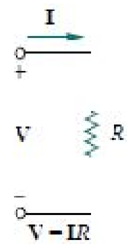
Phasor relationship for circuit elements

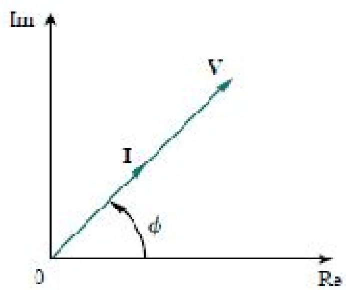
(a) Resistor

$i = I_m \cos(\omega t + \phi)$

$v = i\,R = RI_m \cos(\omega t + \phi)$

$I = I_m\angle\phi \qquad v = R\,I_m\angle\phi$





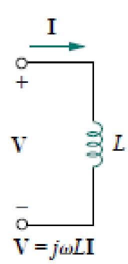
(b) Inductor

$i = I_m \cos(\omega t + \phi)$

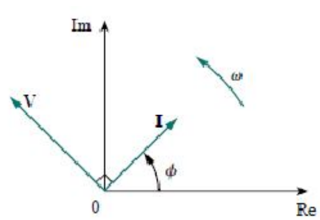
$v = L(di/dt) = -\omega L I_m \sin(\omega t + \phi)$

$v = \omega L I_m \cos(\omega t + \phi + 90^o)$

$I = I_m\angle\phi$ and $V = V_m\angle\phi + 90^o$

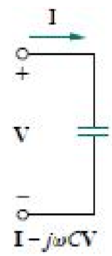


The voltage lead the current by $90^o \rightarrow X_L = j\,\omega\,L$



(b) capacitor

$v(t) = V_m \cos(\omega t+\phi)$ $i = C\, dv/dt$

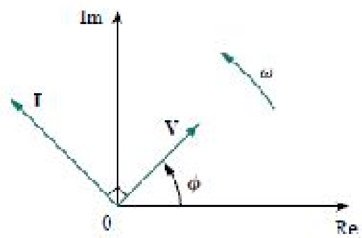


$$i = c\frac{d}{dt}V_m\cos(\omega t+\phi) = -\omega C V_m \sin(\omega t+\phi)$$

$$i = \omega C V_m \cos(\omega t + \phi + 90)$$

$$V = V_m\angle\phi \qquad I = I_m\angle\phi + 90$$

The voltage lag the current by 90° → $X_C = -\,j\, 1/\omega C$



Impedance and Admittance

$$\mathbf{Z} = R + jX = |\mathbf{Z}|\angle\theta \qquad |\mathbf{Z}| = \sqrt{R^2 + X^2}, \qquad \theta = \tan^{-1}\frac{X}{R}$$

$$R = |\mathbf{Z}|\cos\theta, \qquad X = |\mathbf{Z}|\sin\theta$$

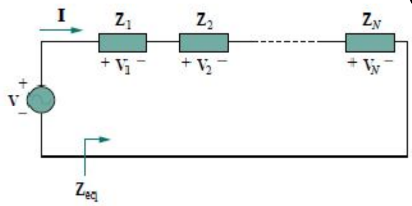
$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{\mathbf{I}}{\mathbf{V}} \qquad \mathbf{Y} = G + jB \qquad G + jB = \frac{1}{R + jX}$$

$$G + jB = \frac{1}{R + jX}\cdot\frac{R - jX}{R - jX} = \frac{R - jX}{R^2 + X^2}$$

$$G = \frac{R}{R^2 + X^2}, \qquad B = \frac{X}{R^2 + X^2}$$

G conductance,   **B susceptance**

Series Circuit

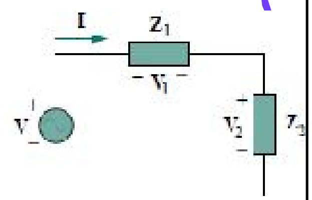


$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_N = \mathbf{I}(\mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N)$$

$$\mathbf{Z}_{eq} = \frac{\mathbf{V}}{\mathbf{I}} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N$$

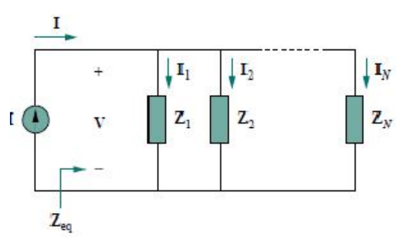
$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_1 + \mathbf{Z}_2}$$

Since $\mathbf{V}_1 - \mathbf{Z}_1\mathbf{I}$ and $\mathbf{V}_2 - \mathbf{Z}_2\mathbf{I}$, then



$$\mathbf{V}_1 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}, \qquad \mathbf{V}_2 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}$$

Parallel Circuit



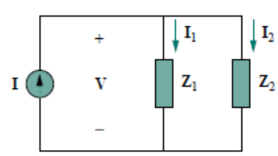
$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_N = \mathbf{V}\left(\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N}\right)$$

$$\frac{1}{\mathbf{Z}_{eq}} = \frac{\mathbf{I}}{\mathbf{V}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N} \qquad \mathbf{Y}_{eq} = \mathbf{Y}_1 + \mathbf{Y}_2 + \cdots + \mathbf{Y}_N$$

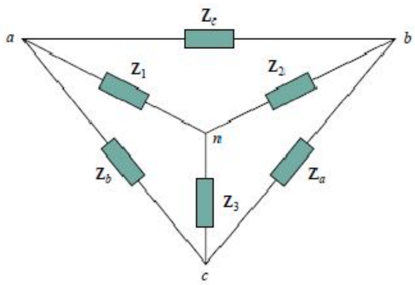
$$\mathbf{Z}_{eq} = \frac{1}{\mathbf{Y}_{eq}} = \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2} = \frac{1}{1/\mathbf{Z}_1 + 1/\mathbf{Z}_2} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}$$

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_{eq} = \mathbf{I}_1\mathbf{Z}_1 = \mathbf{I}_2\mathbf{Z}_2$$

$$\mathbf{I}_1 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}, \qquad \mathbf{I}_2 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}$$



Δ-Y-Conversion



Y-Δ Conversion:

$$\mathbf{Z}_a = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_1}$$

$$\mathbf{Z}_b = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_2}$$

$$\mathbf{Z}_c = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_3}$$

Δ-Y Conversion:

$$\mathbf{Z}_1 = \frac{\mathbf{Z}_b\mathbf{Z}_c}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}$$

$$\mathbf{Z}_2 = \frac{\mathbf{Z}_c\mathbf{Z}_a}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}$$

$$\mathbf{Z}_3 = \frac{\mathbf{Z}_a\mathbf{Z}_b}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}$$

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T v^2(t)dt}$$

$$V_{rms} = \frac{V_m}{\sqrt{2}}$$

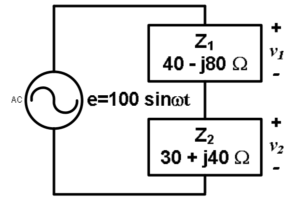
$$V_{av} = \frac{1}{T}\int_0^T v(t)dt$$

Ex.5: consider the circuit,

(a) Calculate the sinusoidal voltages v_1 and v_2 using phasors and voltage divider rule

(b) Sketch the phasor diagram showing E, V_1 and V_2.

(c) Sketch the sinusoidal waveforms of e, v_1 and v_2.



Solution:

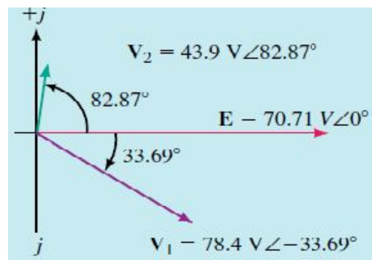
(a) The phasor form of the voltage source is determined as

$$e = 100\sin(\omega t) \Leftrightarrow E = 70.71\angle 0^o\ V$$

$$V_1 = \left(\frac{40 - j80}{(40 - j80) + (30 + j40)}\right)(70.71\angle 0^o) = 78.4\angle -33.69^o V$$

and

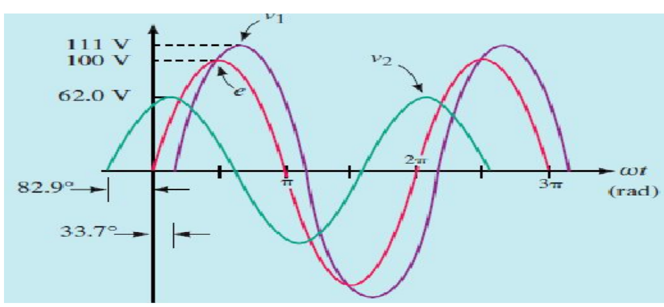
$$V_2 = \left(\frac{30 + j40}{(40 - j80) + (30 + j40)}\right)(70.71\angle 0^o) = 43.9\angle 82.87^o V$$



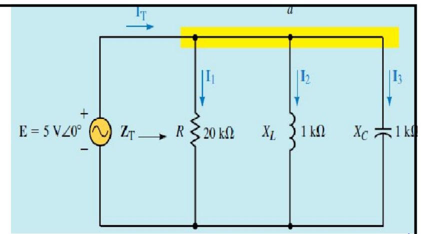
(b) Phasor diagram

(c) $v_1(t) = V_m \sin(\omega t+\phi) = \sqrt{2}x78.4 \sin(\omega t-33.69)$
$= 111 \sin(\omega t-33.69)$

$v_2(t) = V_m \sin(\omega t+\phi) = \sqrt{2}x43.9 \sin(\omega t+82.87)$
$= 62 \sin(\omega t+82.87)$



Ex. 6: Refer to the circuit;
(a) Find the total impedance, Z_T.
(b) Determine the supply current, I_T.
(c) Calculate I_1, I_2, using current divider rule.
(d) Verify Kirchhoff's current law at node a.



Solution

a. $\frac{1}{Z_T} = \frac{1}{20} + \frac{1}{j1} + \frac{1}{-j1} = \frac{1}{20}$ $\qquad Z_T = 20\angle 0^o\ k\Omega$

b. $I_T = \frac{V}{Z_T} = \frac{5\angle 0}{20x10^3\angle 0} = 250x10^{-6}\angle 0^o\ A = 250\angle 0^o\ \mu A$

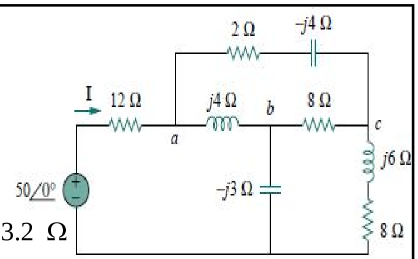
c. $I_1 = \frac{20\angle 0}{20\angle 0} x250\angle 0^o = 250\angle 0^o\ \mu A$

$I_2 = \frac{20\angle 0}{1\angle 90} x250\angle 0^o = 5000\angle -90^o\ \mu A$

$I_3 = \frac{20\angle 0}{1\angle -90} x250\angle 0^o = 5000\angle 90^o\ \mu A$

d. $I_1 + I_2 + I_3 = 250\angle 0 + 5000\angle{-90} + 5000\angle 90 = 250\angle 0^o$
$\therefore$ $I_T = I_1 + I_2 + I_3$ $\therefore$ Kirchhoff's current law is verified.

Ex. 7: Find the current I in the circuit.



Solution:

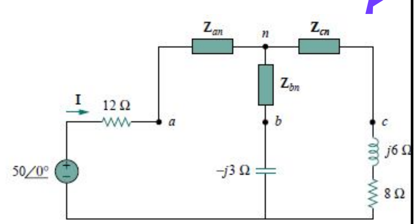
$Z_{an} = \frac{j4(2-j4)}{j4+2-j4+8} = \frac{16+j8}{10} = 1.6 + j0.8\ \Omega$

$Z_{bn} = \frac{j4(8)}{10} = j3.2\ \Omega \qquad Z_{bn} = \frac{8(2-j4)}{10} = 1.6 - j3.2\ \Omega$

$Z_1 = Z_{an}+12 = 13.6 + j0.8$, $Z_2 = Z_{bn} - j3 = j0.2$ and $Z_3 = Z_{cn} + j6+8 = 9.6+j2.8$

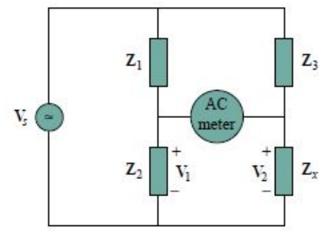
$Z_T = Z_1 + \frac{Z_2 Z_3}{Z_2 + Z_3} = 13.6 + j0.8 + \frac{j0.2(9.6 + j2.8)}{9.6 + j3} = 13.6 + j1\ \Omega$

The source current $I = \frac{V}{Z_T} = \frac{50\angle 0}{13.6 + j1} = 3.666\angle -4.2^o\ \Omega$



AC Bridges

At balance V_1=V_2

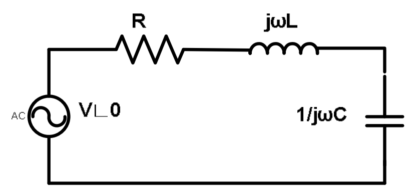


$$V_1 = V_s \frac{Z_2}{Z_1 + Z_2} \qquad V_2 = V_s \frac{Z_x}{Z_3 + Z_x}$$

$$\frac{Z_2}{Z_1 + Z_2} = \frac{Z_x}{Z_3 + Z_3}$$

$$\therefore Z_x = \frac{Z_3}{Z_1} Z_2$$

Series Resonance



$$Z = R + j(\omega L - \frac{1}{\omega C})$$

At resonance $\mathrm{Im}(z) = 0$

$$\omega L - \frac{1}{\omega C} = 0 \qquad \omega_o L = \frac{1}{\omega_o C}$$

$$\omega_o = \frac{1}{\sqrt{LC}} \; rad/s \qquad f_o = \frac{1}{2\pi\sqrt{LC}} \; Hz$$

$$|I| = \frac{V}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}$$

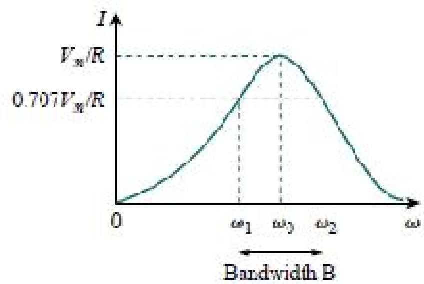
The highest power is at ω_o

$$P(\omega_o) = I^2 R = \frac{V^2}{R}$$

The half power frequencies are at:

$$I = \frac{I}{\sqrt{2}} = 0.707I$$

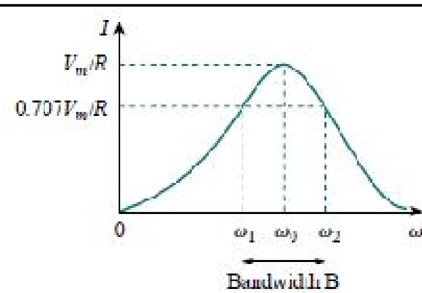
$$P(\omega_1) = P(\omega_2) = \frac{1}{2}\frac{V^2}{R}$$



ω_1 & ω_2 can be get by setting $Z = \sqrt{2}R$

$$\sqrt{R^2 + (\omega L - \frac{1}{\omega C})} = \sqrt{2}R$$

$$\sqrt{R^2 + (\omega L - \frac{1}{\omega C})} = \sqrt{2}R$$



By solving the above equation for ω

$$\omega_1 = \frac{-R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

Bandwidth "It is defined as the half power bandwidth"
B = $\omega_2 - \omega_1$ = (R / L)

$$\omega_o = \sqrt{\omega_1 \omega_2}$$

Quality Factor Q

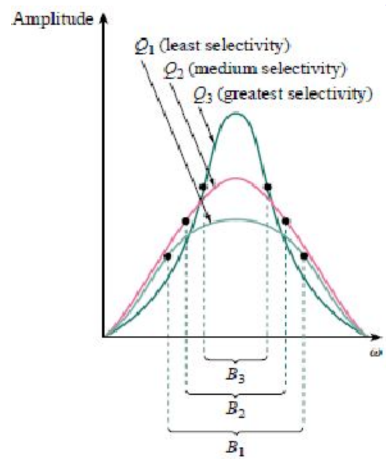
"Measure the sharpness of resonance "

$$Q = 2\pi \frac{\text{Peak energy stored in the circuit}}{\text{Energy dissipated by the circuit}}$$

$$Q = 2\pi \frac{\frac{1}{2}LI^2}{\frac{1}{2}I^2R(1/f)} = \frac{2\pi f L}{R}$$

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 C R}$$

$$B = \frac{R}{L} = \frac{\omega_0}{Q}$$

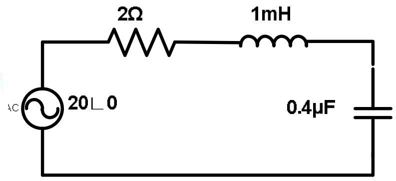


A resonant circuit is designed to operate at or near its resonant frequency.

For high-Q circuits ($Q \geq 10$)

$$\omega_1 \simeq \omega_0 \quad \frac{R}{2}, \qquad \omega_2 \simeq \omega_0 + \frac{R}{2}$$

Example: For the circuit shown:
(a) Find the resonant frequency and the half power frequencies
(b) Calculate the quality factor and bandwidth
(c) Determine the amplitude of the current at ω_o, ω_1 and ω_2



Solution

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10^{-3} \times 0.4 \times 10^{-6}}} = 50 \text{ krad/s}$$

$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$= -\frac{2}{2 \times 10^{-3}} + \sqrt{(10^3)^2 + (50 \times 10^3)^2}$$

$$= -1 + \sqrt{1 + 2500} \text{ krad/s} = 49 \text{ krad/s}$$

$$\omega_2 = 1 + \sqrt{1 + 2500} \text{ krad/s} = 51 \text{ krad/s}$$

$$Q = \frac{\omega_0}{B} = \frac{50}{2} = 25$$

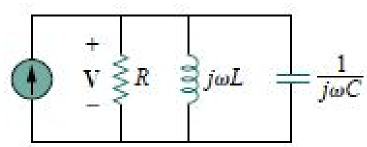
$$B = \omega_2 - \omega_1 = 2 \text{ krad/s}$$

At $\omega = \omega_0$, $I = \frac{V_m}{R} = \frac{20}{2} = 10 \text{ A}$

At $\omega = \omega_1, \omega_2$, $I = \frac{V_m}{\sqrt{2}R} = \frac{10}{\sqrt{2}} = 7.071 \text{ A}$

Parallel Resonance

$$\mathbf{Y} = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$$



At resonance Im(Y)=0

$$\omega C - \frac{1}{\omega L} = 0 \qquad \omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}$$

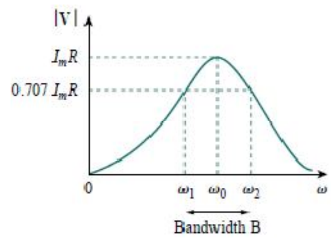
At resonance:
(1)The parallel LC combination acts like open circuit. i.e. the entire current flows through R
(2) The inductor and capacitor currents can be much more than the source current

$$\omega_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

$$\omega_2 = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

$$B - \omega_2 - \omega_1 - \frac{1}{RC}$$

$$Q = \frac{\omega_0}{B} = \omega_0 RC = \frac{R}{\omega_0 L}$$

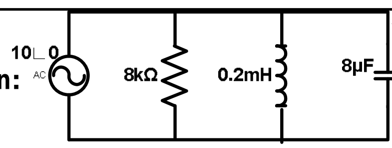


$$\omega_1 - \omega_0\sqrt{1 + \left(\frac{1}{2Q}\right)^2} - \frac{\omega_0}{2Q}, \qquad \omega_2 - \omega_0\sqrt{1 + \left(\frac{1}{2Q}\right)^2} + \frac{\omega_0}{2Q}$$

for high-Q circuits ($Q \geq 10$)

$$\omega_1 \simeq \omega_0 - \frac{B}{2}, \qquad \omega_2 \simeq \omega_0 + \frac{B}{2}$$

Example: **For the circuit shown:**



(a) Calculate ω_o, Q and B
(b) Find ω_1 and ω_2
(c) Determine the power dissipated at ω_o, ω_1 and ω_2

Solution

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.2 \times 10^{-3} \times 8 \times 10^{-6}}} = \frac{10^5}{4} = 25 \text{ krad/s}$$

$$Q = \frac{R}{\omega_0 L} = \frac{8 \times 10^3}{25 \times 10^3 \times 0.2 \times 10^{-3}} = 1600$$

$$B = \frac{\omega_0}{Q} = 15.625 \text{ rad/s}$$

$$\omega_1 = \omega_0 \quad \frac{B}{2} = 25{,}000 \quad 7.812 = 24{,}992 \text{ rad/s}$$

$$\omega_2 = \omega_0 + \frac{B}{2} = 25{,}000 + 7.8125 = 25{,}008 \text{ rad/s}$$

At $\omega = \omega_0$, $\mathbf{Y} = 1/R$ or $\mathbf{Z} = R = 8\ \text{k}\Omega$.

$$\mathbf{I}_o = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{10\angle -90^\circ}{8000} = 1.25\angle -90^\circ\ \text{mA}$$

at $\omega = \omega_0$ $\quad P = I_o^2 R = (1.25x10^{-3})^2(8000) = 12.5x10^{-3} W$

At $\omega = \omega_1,\ \omega_2$: $\quad P = \dfrac{V^2}{2R} = \dfrac{10^2}{2x8000} = 6.25x10^{-3} W$